5.1 Eigenvectors and Eigenvalues

Consider the

matrix $A = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}$

and the vectors $\hat{u} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and $\hat{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Notice

$$A\hat{u} = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} = 1.\hat{u}$$

and

$$A\hat{v} = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 4.\hat{v}$$

In general, $A\hat{\omega} \neq \lambda\hat{\omega}$, $\lambda \in \mathbb{R}$ (Pick an arbitrary vector)

Thus, $\hat{u}$ and $\hat{v}$ are special vectors.

Also, any multiple of them has the same property.

In fact,

$$\begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ -4 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \end{bmatrix}, \quad \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 5 \\ 5 \end{bmatrix} = \begin{bmatrix} 20 \\ 20 \end{bmatrix} = 4.\begin{bmatrix} 5 \\ 5 \end{bmatrix}$$

$$= 1.\begin{bmatrix} 2 \\ -4 \end{bmatrix}$$

In general, for any $\alpha \in \mathbb{R}$.

For $\vec{u} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$, $A(\alpha \vec{u}) = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} \alpha \\ -2\alpha \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \alpha \vec{u}$

For $\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $A(\alpha \vec{v}) = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} \alpha \\ \alpha \end{bmatrix} = \alpha \begin{bmatrix} 4 \\ 4 \end{bmatrix} = 4(\alpha \vec{v})$

The vectors $\vec{u}$ and $\vec{v}$ are called <u>eigenvectors of matrix A</u>, an the scalars $\lambda_1 = 1$, $\lambda_2 = 4$ are called <u>eigenvalues of A</u>.

<u>Important Question</u>:

Given a matrix $A_{n \times n}$

How can we find its eigenvalues and eigenvectors?

or Find λ and $\vec{x}$ such that

$$\boxed{A\vec{x} = \lambda \vec{x}} \qquad (2.1)$$

Equation (2.1) is equivalent to

$$A\vec{x} - \lambda\vec{x} = \vec{0} \quad \text{or} \quad A\vec{x} - \lambda I\vec{x} = \vec{0}$$

or $$\boxed{(A - \lambda I)\vec{x} = \vec{0}} \qquad (3.1)$$

If we call $B = A - \lambda I$

Equation (3.1) is equivalent to

$$\boxed{B\vec{x} = \vec{0}} \qquad (3.2)$$

Obviously, $\vec{x} = \vec{0}$ is always a solution of (3.1) for any A and for any λ.

We are interested in finding any λ and $\vec{x} \neq \vec{0}$ Such that equation (3.1) be satisfied.

It means we are looking for λ which gives a nontrivial solution for (3.1).

What condition should we impose on $B = A - \lambda I$ for the existence of nontrivial solutions?

$$\boxed{\det(B) = \det(A - \lambda I) = 0} \qquad (3.3)$$

This condition leads to an algebraic equation for λ.

This suggests a procedure for computing λ and $\vec{x}$.

Back to our example to illustrate the procedure.

If $A = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}$, then

$$\det(A-\lambda I) = |A - \lambda I| = \left| \begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right| = \begin{vmatrix} 3-\lambda & 1 \\ 2 & 2-\lambda \end{vmatrix}$$

$$= (3-\lambda)(2-\lambda) - 2 =$$
$$= (\lambda-3)(\lambda-2) - 2 = \lambda^2 - 5\lambda + 6 - 2$$
$$= \lambda^2 - 5\lambda + 4 = (\lambda-1)(\lambda-4)$$

Therefore, $|A - \lambda I| = 0$ if $\lambda_1 = 1$, $\lambda_2 = 4$.

Now, we will find the eigenvectors associated to each one of the eigenvalues.

For: $\lambda_1 = 1$, we have to solve the homogeneous system $(A - 1.I)\vec{x} = \vec{0}$ (11.1)

We know, this system has non-trivial solutions Why?.

or

$$\begin{bmatrix} 3-1 & 1 \\ 2 & 2-1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or

$$\begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \begin{cases} 2v_1 + v_2 = 0 \\ 2v_1 + v_2 = 0 \end{cases}$$

$$\begin{pmatrix} 2 & 1 & 0 \\ 2 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{array}{l} 2v_1 + v_2 = 0 \\ v_2 = -2v_1 \end{array}$$

or $\vec{v} = t\begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} t \\ -2t \end{pmatrix}$ for any $t \neq 0$.

This is a family of eigenvectors associated to the eigenvalue $\boxed{\lambda_1 = 1}$.

In particular $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ $\begin{pmatrix} 2 \\ -4 \end{pmatrix}$ are eigenvectors of $\lambda = 1$

We will do the same work to find the eigenvectors for $\boxed{\lambda_2 = 4}$

$$\begin{bmatrix} 3-4 & 1 \\ 2 & 2-4 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Leftrightarrow \begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

row-reducing

$$\begin{pmatrix} -1 & 1 & 0 \\ 2 & -2 & 0 \end{pmatrix} \sim \begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \begin{cases} -w_1 + w_2 = 0 \\ 2w_1 - 2w_2 = 0 \end{cases}$$

$\Rightarrow -w_1 + w_2 = 0 \Rightarrow w_2 = w_1$ or $\vec{w} = t\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, for any $t \neq 0$.

or $\vec{w} = \begin{pmatrix} t \\ t \end{pmatrix}$

In particular, $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 4 \\ 4 \end{pmatrix}$ are eigenvectors of $\lambda = 4$

5.1 6) a)

$$A=\begin{bmatrix}4&0&1\\-2&1&0\\-2&0&1\end{bmatrix}$$

$$A-\lambda I = \begin{vmatrix}4-\lambda & 0 & 1\\ -2 & 1-\lambda & 0\\ -2 & 0 & 1-\lambda\end{vmatrix} = 0 \Leftrightarrow -(\lambda-4)(\lambda-1)(\lambda-1)+2(\lambda-1)=0$$

$$\Leftrightarrow -(\lambda-1)\left[(\lambda-4)(\lambda-1)+2\right]=0$$

$$\Leftrightarrow -(\lambda-1)\left[\lambda^2-5\lambda+4+2\right] = -(\lambda-1)\left[\lambda^2-5\lambda+6\right]=0$$

$$\Leftrightarrow (\lambda-1)\left[\lambda^2-5\lambda+6\right]=0 \Leftrightarrow \lambda^3-\lambda^2-5\lambda^2+5\lambda+6\lambda-6=0$$

$$\Leftrightarrow \lambda^3-6\lambda^2+11\lambda-6=0$$

or $(\lambda-1)(\lambda^2-5\lambda+6)=0 \Leftrightarrow (\lambda-1)(\lambda-3)(\lambda-2)=0$

Def. $\boxed{P(\lambda)=\lambda^3-6\lambda^2+11\lambda-6}$ is called characteristic polynomial

and $\boxed{\lambda^3-6\lambda^2+11\lambda-6=0}$ is called characteristic equation.

In general,

$|A-\lambda I|=0$ is called characteristic equ. of matrix A

and $\boxed{P(\lambda)=|A-\lambda I|}$ is the characteristic polynomial.

In case that you arrive to the characteristic equation

$$\boxed{\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0} \qquad (*)$$

How do you obtain the eigenvalues ?

Remark: Given a polynomial: $\boxed{\lambda^n + c_1\lambda^{n-1} + \dots + c_n = 0}$

with integer coefficients, all integer solutions (if any) must be a divisor of the constant term c_n.

In our case, equation (*) has $c_n = 6 \Rightarrow$ Divisors $\pm 1, \pm 2, \pm 3$

In particular,

$$(1)^3 - 6(1)^2 + 11(1) - 6 = 0 \checkmark \qquad \lambda_1 = 1$$

$$\Rightarrow \begin{array}{l|l} \cancel{\lambda^3} - 6\lambda^2 + 11\lambda - 6 & \lambda - 1 \\ \hline -\cancel{\lambda^3} + \lambda^2 & \lambda^2 - 5\lambda + 6 \\ \hline -5\cancel{\lambda^2} + 11\lambda - 6 & \\ +\cancel{5}\lambda^2 - 5\lambda & \\ \hline 6\lambda - 6 & \\ -6\lambda + 6 & \\ \hline 0 & \end{array}$$

then,

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = (\lambda - 1)(\lambda^2 - 5\lambda + 6)$$
$$= (\lambda - 1)(\lambda - 3)(\lambda - 2) \checkmark$$

To obtain eigenvectors corresponding to eigenvalue $\lambda_1 = 1$

$(A - 1I)\vec{x} = \vec{0} \iff$

$$\begin{bmatrix} 3 & 0 & 1 \\ -2 & 0 & 0 \\ -2 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{array}{l} 3x_1 + x_3 = 0 \\ -2x_1 = 0 \Rightarrow x_1 = 0 \\ \text{and} \quad x_3 = -3x_1 = 0 \end{array}$$

$$\Rightarrow \quad \vec{x} = \begin{pmatrix} 0 \\ x_2 \\ 0 \end{pmatrix} = x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Actually $$A\hat{x} = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \checkmark$$

Similar for the others. do it!

The Set of all eigenvectors for a given λ plus the vector $\vec{0}$ form a Subspace. It's actually the null$(A - \lambda I)$

This Subspace is called eigenspace Corresponding to λ.

In our previous problem,

$$\underline{\text{Eigenspace of } \lambda_1 = 1} = \left\{ \vec{x} : \ \vec{x} = \alpha \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \ \alpha \in \mathbb{R} \right\}$$

It contains all eigenvectors corresp. to λ plus $\vec{x} = \vec{0}$, which is not an eigenvector

Thm - λ eigenvalue of A ($n \times n$ matrix)

- $\vec{x}$ corresp. eigenvector

$$\Rightarrow \quad A^k \vec{x} = \lambda^k \vec{x}, \quad k = 1, 2, \ldots$$

Proof.- If $A\vec{x} = \lambda\vec{x} \Rightarrow A^2\vec{x} = A(A\vec{x}) = A(\lambda\vec{x}) = \lambda(A\vec{x}) = \lambda(\lambda\vec{x}) = \lambda^2\vec{x}$. repeat this by induction.

Eigenvalues and Eigenvectors

Def. - A $n \times n$

- $\vec{x} \neq \vec{0} \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$

If $A\vec{x} = \lambda \vec{x}$, then λ is called an eigenvalue of A, and $\vec{x}$ is called the eigenvector corresp. to λ.

Thm. - A $n \times n$

λ is an eigenvalue of A $\overset{\text{if and only if}}{\Longleftrightarrow}$ $\det(A - \lambda I) = 0$

Proof.

λ is an eigenv. of A

$\Updownarrow$

There exists $\vec{x} \neq \vec{0}$ s.t. $A\vec{x} = \lambda \vec{x} \Leftrightarrow A\vec{x} - \lambda \vec{x} = \vec{0} \Leftrightarrow (A - \lambda I)\vec{x} = \vec{0}$

or $B\vec{x} = \vec{0}$ and $\vec{x} \neq \vec{0}$.

From invertible matrix theorem

$B\vec{x} = (A - \lambda I)\vec{x} = \vec{0}$ has a nontrivial solution

$\Longleftrightarrow \det(A - \lambda I) = 0$ ✓

An immediate consequence of the previous theorem is

Corollary $A_{n\times n}$ and $\lambda \in \mathbb{R}$

λ is an eigenvalue of $A \Leftrightarrow$ the equation

$$(A-\lambda I)\vec{x}=\vec{0} \qquad (7.1)$$

has a nontrivial Solution.

Def.- the set of all solutions of (7.1) is called the eigenspace of A Corresponding to λ.

Notice that the eigenspace of A is Nul $(A-\lambda I)$ which is a Subspace of $\mathbb{R}^n$. It contains $\vec{x}=\vec{0}$ which is not an eigenvector.

Corollary: $A_{n\times n}$

$\lambda=0$ is an eigenvalue of A $\Leftrightarrow$ A is not invertible.

Proof.-

$\lambda=0$ is an eigenvalue of A $\Leftrightarrow$ $A\vec{x}=\vec{0}$ has nontrivial Solutions $\Leftrightarrow$ A is not invertible. (from Inv. Matrix Thm).

What happens if A is a triangular matrix?

$$A = \begin{bmatrix} 3 & 1 & 2 \\ 0 & 2 & -5 \\ 0 & 0 & 7 \end{bmatrix} \Rightarrow A - \lambda I = \begin{bmatrix} 3-\lambda & 1 & 2 \\ 0 & 2-\lambda & -5 \\ 0 & 0 & 7-\lambda \end{bmatrix}$$

$$|A - \lambda I| = (3-\lambda)(2-\lambda)(7-\lambda) = 0 \Leftrightarrow \begin{array}{l} \lambda_1 = 3 \\ \lambda_2 = 2 \\ \lambda_3 = 7 \end{array}$$

<u>Thm 1</u>

The eigenvalues of a triangular matrix are the entries on its main diagonal.

<u>Proof</u>.- Similar to previous example. Use Cofactor expansion along 1^{st} Column (upper triang.) or 1^{st} row (Lower triangular) and ...

<u>Thm 2</u> If $\vec{v}_1, .., \vec{v}_r$ are eigenvectors corresponding to distinct eigenvalues $\lambda_1, .., \lambda_r$ of an $n \times n$ matrix A, then the set $\{\vec{v}_1, .., \vec{v}_r\}$ is lin. indep.

Remark: Observe in the previous example

$$A = \begin{bmatrix} 3 & 1 & 2 \\ 0 & 2 & -5 \\ 0 & 0 & 7 \end{bmatrix}$$

Eigenvalues are $\lambda_1 = 3,\ \lambda_2 = 2,\ \lambda_3 = 7$

and $\det(A) = 3 \cdot 2 \cdot 7 = \lambda_1 \cdot \lambda_2 \cdot \lambda_3$

this property is true in general for any matrix A but we need to consider eigenvalues repeated according to their multiplicities and also complex eigenvalues if any. (Problem 19 Sect. 5.2).

New addition to invertible matrix thm.

Thm $A_{n\times n}$. A invertible if and only if

a) $\lambda = 0$ is not an eigenvalue of A

s) $\det(A) \neq 0$

Proof.- $\boxed{(a) \Rightarrow (s) \Rightarrow A \text{ inv.}}$

a) $\Rightarrow$ s) $\lambda = 0$ is an eigenvalue of A if and only if $\det(A - \overset{=0}{\lambda} I) = 0 \Leftrightarrow \det(A) = 0$

or $\lambda = 0$ is not an eigenvalue of A if and only if $\det(A) \neq 0$.

(s) $\Rightarrow$ (a)

Recall the following fact:

For $A_{n\times n}$ a row reduction to a matrix U only by row replacements and row interchanges without row replacements is always possible.
If there are "r" interchanges

$$A = \begin{bmatrix} a_{11} & a_{12} \cdots & a_{1n} \\ a_{21} & a_{22} \cdots & a_{2n} \\ \vdots & & \\ a_{n1} & a_{n2} \cdots & a_{nn} \end{bmatrix} \sim U = \begin{bmatrix} u_{11} & u_{12} \cdots & u_{1n} \\ 0 & u_{22} \cdots & u_{2n} \\ \vdots & & \\ 0 & \cdots\ 0 & u_{nn} \end{bmatrix}$$

and

$$\det A = (-1)^r \det U = (-1)^r (u_{11})(u_{22})\cdots(u_{nn})$$

A is not invertible if and only if

$\det U = 0$ if and only if $\det A = 0$.

Then $\det(A) \neq 0 \Leftrightarrow A$ is invertible.

Similarity

Def. $A_{n\times n}$ is similar to $B_{n\times n}$

if and only if there exists P such that

$$P^{-1}AP = B$$

$$T: S \subset M_{n\times n} \longrightarrow M_{n\times n} \qquad \text{Similarity transf.}$$
$$A \longrightarrow P^{-1}AP$$

thm 4 If $A_{n\times n}$ and $B_{n\times n}$ are similar then, they have the same charact. Polyn.
and same eigenvalues.

Proof.

If $B = P^{-1}AP$, then

$$B - \lambda I = P^{-1}AP - \lambda I = P^{-1}AP - \lambda P^{-1}P =$$

$$= P^{-1}(AP - \lambda P) = P^{-1}(A - \lambda I)P$$

$$\Rightarrow \det(B - \lambda I) = \det\left(P^{-1}(A-\lambda I)P\right) =$$

$$\det(P^{-1})\det(A-\lambda I)\det(P) = \det(\underbrace{PP^{-1}}_{=I})\,\det(A-\lambda I)$$

$$= 1 \cdot \det(A - \lambda I) \checkmark.$$